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THEORY OF COHESIVE CRACK MODEL WITH INTERACTIVE CRACKS

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Abstract—A theory of cohesive crack model is proposed to study crack interaction. The elastic behavior of the structure is represented by the influence functions, and the model is cast as integral equations. When there is more than one crack, the behavior of unloading cracks must be studied. The stability property of the cohesive crack model is characterized by its rate equations. As an example, the theory is applied to solve the problem of a half plane with periodic cracks on its surface. Some interesting features of the solution are described. © 1998 Elsevier Science Ltd.

1. INTRODUCTION

When a structure contains more than one crack, the question of how cracks interact must be answered. These cracks may grow simultaneously, or some cracks may close during certain stages of crack growth. In the framework of linear fracture mechanics, Bazant (1977) and Nemat-Nasser *et al.* (1978) studied the interaction of equidistant parallel surface cracks. In the same framework, Nguyen (1987) summarized the crack interaction problem into a special case in the more general framework of the stability theory of dissipative media. The problem of cohesive crack model with multiple interactive cracks has not been studied.

The theory of cohesive crack model (CCM) to be discussed in this paper is a nonlinear fracture mechanics theory. Although Barenblatt (1962) first attributed the crack resistance toughness to the residual strength (cohesive stress) of the material, he restricted himself to the case of brittle material where the process zone is very small and the cohesive stress distribution is independent of the crack opening displacement. Dugdale (1960) and Bilby *et al.* (1963) restricted themselves to the case of uniform constant cohesive stress, though there is no restriction on process zone sizes. These are two examples of what we might call linear cohesive crack models. Hillerborg *et al.* (1976) introduced a stress-separation relation (softening law) to describe the distribution of the cohesive stress in a finite process zone, and therefore the cohesive crack model becomes nonlinear. Li and Liang (1993) developed a theory of CCM in which the peak load of the Griffith problem was solved through the condition of stability limit, which was transformed into an eigenvalue problem under the assumption of linear softening law. Using the same technique, Li and Hong (1992) solved the problems of double-notched or center-notched infinite strips under remote tensile loading.

In this paper the theory of CCM with interactive cracks is formulated. First, the process zone equations and the crack tip equations are established as a simple generalization from the single-crack case. These equations are called the basic equations of CCM. To complete the formulation, one must also consider the unloading behavior of cohesive cracks. Based on the rate form of the basic equations, the critical condition of CCM with interactive cracks can be reduced to an eigenvalue problem. For illustrative purposes, the problem of parallel cracks on the surface of a half plane subjected to shrinkage loading is

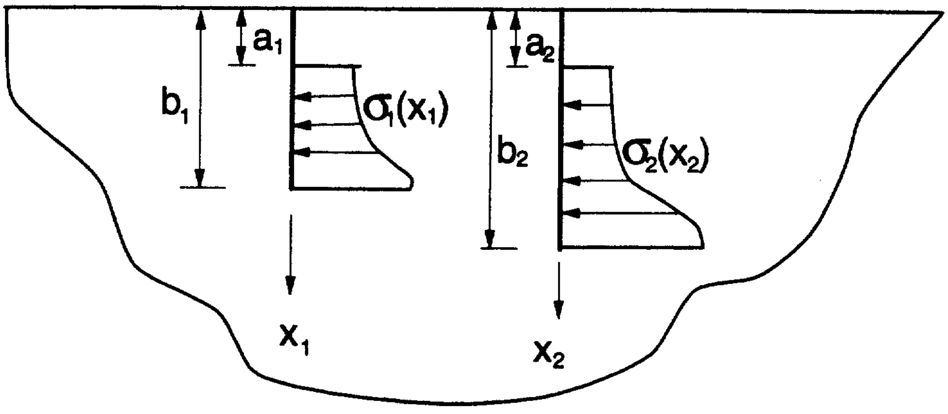

Fig. 1. Schematic of two interactive cohesive cracks.

solved. it is demonstrated that the initial state of uniform growth of cracks loses its uniqueness, and the stable uniform growth mode yields to nonuniform crack growth, where every another crack stop growing and start to close.

2. BASIC EQUATIONS OF COHESIVE CRACK MODEL WITH TWO CRACKS

Utilizing the compliance influence functions (Green's function), we can express the crack opening displacements as

$$
\begin{aligned}
w_1(x_1) &= -\int_{a_1}^{b_1} C_{11}^{\sigma\sigma}(x_1, x_{1'})\sigma_1(x_{1'})\,\mathrm{d}x_{1'} - \int_{a_2}^{b_2} C_{12}^{\sigma\sigma}(x_1, x_{2'})\sigma_2(x_{2'})\,\mathrm{d}x_{2'} + C_1^{\sigma P}(x_1)P \\
w_2(x_2) &= -\int_{a_1}^{b_1} C_{21}^{\sigma\sigma}(x_2, x_{1'})\sigma_1(x_{1'})\,\mathrm{d}x_{1'} - \int_{a_2}^{b_2} C_{22}^{\sigma\sigma}(x_2, x_{2'})\sigma_2(x_{2'})\,\mathrm{d}x_{2'} + C_2^{\sigma P}(x_2)P \qquad (1)
\end{aligned}
$$

where x_i ($i = 1, 2$) are the coordinates measured along crack one and crack two, as shown in Fig. 1. a_i and b_i are the initial notch coordinates and the process zone tip coordinators for each crack. w_i = crack opening displacements.

To clarify possible confusion, we want to point out that eqn (1) applies only to situations where geometric configuration guarantees that cracks grow in their own planes. If symmetry cannot hold for both cracks, as is implied in Fig. 1, crack surfaces slippage as well as shear stresses, in addition to crack opening and cohesive stress, must be taken into account. However, such considerations are precluded by the scope of this paper. The theory of cohesive crack model with a mode II component will be pursued in a future study.

The compliance functions are denoted by C. $C_i^{\sigma P}(x_i)$ = crack opening displacement at x_i due to a unit load P, and $C_{ij}^{\sigma\sigma}(x_i, x_j)$ = crack opening displacement at x_i due to a pair of unit forces acting on the crack surface at position x_j. Due to the assumption of linear elasticity in the bulk material, $C_{ij}^{\sigma\sigma}(x_i, x_j)$ is symmetric with respect to i and j. P = load parameter.

The cohesive stress is related to crack opening displacement by the softening law (Fig. 2)

$$w = g(\sigma) \qquad (2)$$

When $w = 0$, $\sigma = f_t$ = tensile strength of the material; and when $w = w_c$ = crack opening threshold value then $\sigma = 0$, this means the material is completely severed. Substituting (2) into (1), we obtain the process zone equation as

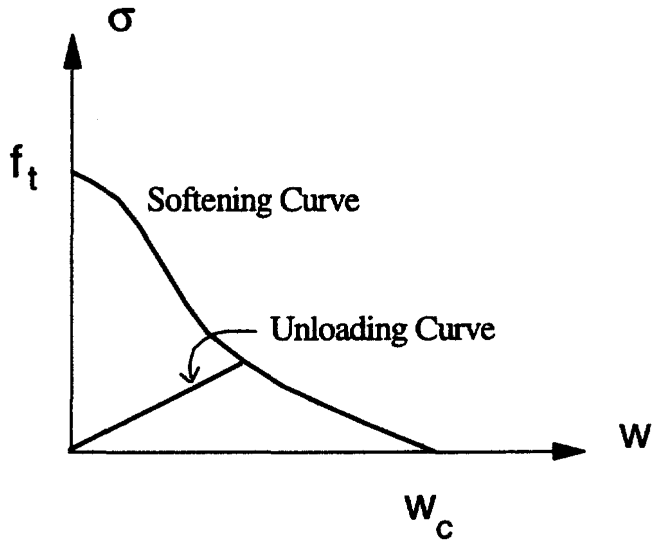

Fig. 2. Softening law with an unloading rule.

$$G_i = g[\sigma(x_i)] + \int_{a_j}^{b_j} C_{ij}^{\sigma\sigma}(x_i, x')\sigma_j(x')\,\mathrm{d}'x - C_i^{\sigma P}(x_i)P = 0 \tag{3}$$

where $i = 1, 2$. The summation convention utilized in this paper observe the following rule: the repeated subscript (dummy index), no matter how many times, in multiplication or division implies summation over its proper range, unless specified otherwise. However, the repeated subscript in the argument or in the integral-limit variables does not imply summation. With this convention, the expressions can be drastically simplified. The total stress intensity factor in each crack tip must be zero, which can be expressed as the conditions

$$\begin{aligned} k_1^P P - \int_{a_1}^{b_1} k_{11}^{\sigma}(x)\sigma_1(x)\,\mathrm{d}x - \int_{a_2}^{b_2} k_{12}^{\sigma}(x)\sigma_2(x)\,\mathrm{d}x = 0 \\ k_2^P P - \int_{a_1}^{b_1} k_{21}^{\sigma}(x)\sigma_1(x)\,\mathrm{d}x - \int_{a_2}^{b_2} k_{22}^{\sigma}(x)\sigma_2(x)\,\mathrm{d}x = 0 \end{aligned} \tag{4}$$

where k_i^P $(i = 1, 2)$ = stress intensity factor at the tip of crack i due to a unit force P, and $k_{ij}^{\sigma}(x_j)$ = stress intensity factor at the tip of crack i due to a pair of unit forces on the crack surfaces at the position x_j. As was pointed out by Barenblatt (1962) and Schapery (1975), when the total stress intensity factor is zero, the crack tip stress in the material is equal to the tensile strength of the material. Equation (4) can be simplified by using the summation convention as

$$K_i = k_i^P P - \int_{a_j}^{b_j} k_{ij}^{\sigma}(x)\sigma_j(x)\,\mathrm{d}x = 0 \tag{5}$$

Equations (3) and (5) are the basic equations of CCM with two cracks. If there are more than two cracks, one can simply extend the range of indexes. The basic equations of CCM as expressed by (3) and (5) can be generally written as

$$F_i(\sigma_j, b_j; P) = 0 \tag{6}$$

If the range of j is from 1 to n, then $i = 1, 2, \ldots, 2n$, so there are just enough number of equations to solve the unknown stress distribution σ_j and crack length b_j. It is sometimes useful to call $q_i = (\sigma_i, b_i)$ the state variables, for the system of structure with cracks is totally determined once the state variables are specified. The load P is taken as a parameter. For a given value of P, we might be able to determine the corresponding state variables, but uniqueness and existence of the solution are not always guaranteed. There may exist only one solution for a given value of P, or more than one solution for another value of P, or there may be no solution at all. The major objective of this paper is to see how the solutions respond to the changing loading parameter.

3. RATE EQUATIONS OF CCM

To study the behavior of solutions, one needs to study the rate form of the basic equations. The rate equation can be obtained by considering the basic unknown variables as well as the loading parameter to be the function of time t, which is only for the purpose of keeping the sequences of the system development. When b_i changes, the compliance functions also change. The derivatives of the compliance functions with crack length b_i can be expressed in terms of stress intensity factors

$$\frac{\partial C_{ij}^{\sigma\sigma}(x_i, x_j)}{\partial b_m} = \frac{2}{E'} k_{mi}^{\sigma}(x_i) k_{mj}^{\sigma}(x_j)\,; \quad \frac{\partial C_i^{\sigma P}(x_i)}{\partial b_m} = \frac{2}{E'} k_{mi}^{\sigma}(x_i) k_m^P. \tag{7}$$

For these expressions no summation over m is implied. $E' = E$ (the Young's modulus) for plane stress condition and $E' = E/(1-v^2)$ for plain strain condition, v = Poisson's ratio. This type of relations can be derived under the assumption of linear elasticity and was obtained, for instance, by Okamura (1975). Equation (7) is simply a generalization to the case of multiple cracks.

The time derivative of (3) can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t} G_i(\sigma_j, b_j\,; P) = g'\dot{\sigma}_i(x_i) + \int_{a_j}^{b_j} C_{ij}^{\sigma\sigma}(x_i, x)\dot{\sigma}_j(x)\,\mathrm{d}x - C_i^{\sigma P}(x_i)\dot{P}$$
$$+\dot{b}_m \int_{a_j}^{b_j} \frac{\partial C_{ij}^{\sigma\sigma}(x_i, x)}{\partial b_m}\sigma_j(x)\,\mathrm{d}x - \dot{b}_m \frac{\partial C_i^{\sigma P}(x_i)}{\partial b_m} P + \dot{b}_j C_{ij}^{\sigma\sigma}(x_i, b_j)\sigma_j(b_j) = 0 \tag{8}$$

where a dot denotes a derivative with respect to time t. As conventional, g' denotes the derivative of softening function g with respect to its argument σ. The last term comes from the derivative with respect to the upper integral limit. However, at crack tip, the compliance function is zero, thus the last term can be dropped. The next last two terms can be simplified by taking into account of (7) and (5)

$$\dot{b}_m \int_{a_j}^{b_j} \frac{\partial C_{ij}^{\sigma\sigma}(x_i, x)}{\partial b_m}\sigma_j(x)\,\mathrm{d}x - \dot{b}_m \frac{\partial C_i^{\sigma P}(x_i)}{\partial b_m} P$$
$$= \frac{2}{E'} \dot{b}_m k_{mi}^{\sigma}(x_i) \left(\int_{a_j}^{b_j} k_{mj}(x)\sigma_j(x)\,\mathrm{d}x - k_m^P P \right) = -\frac{2}{E'} \dot{b}_m k_{mi}^{\sigma}(x_i) K_m. \tag{9}$$

Thus, eqn (8) can always be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} G_i = g'\dot{\sigma}_i(x_i) + \int_{a_j}^{b_j} C_{ij}^{\sigma\sigma}(x_i, x)\dot{\sigma}_j(x)\,\mathrm{d}x - C_i^{\sigma P}(x_i)\dot{P} - \frac{2}{E'} k_{mi}^{\sigma}(x_i)\dot{b}_m K_m = 0. \tag{10}$$

The time derivative of (5) can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}K_i = k_i^P \dot{P} - \int_{a_j}^{b_j} k_{ij}^\sigma(x)\dot{\sigma}_j(x)\,\mathrm{d}x + \dot{b}_m \frac{\partial k_i^P}{\partial b_m} P - \dot{b}_m \frac{\partial}{\partial b_m}\int_{a_j}^{b_j} k_{ij}^\sigma(x)\sigma_j(x)\,\mathrm{d}x = 0. \tag{11}$$

In calculating the last term, we deliberately avoid expanding the derivative into the sum of the derivative with respect to upper integral limit and the derivative of k_{ij}^σ with respect to b_m, because each term will be unbounded with the order of $r^{-1/2}$ (as $r \to 0$). However, since we expect the final result to be bounded, these singular terms must cancel each other to yield a bounded value.

Equations (10) and (11) are the rate equations of CCM, which depend on the state variable q_i, but are linear in the rate of state variables and loading rate. For a given loading rate P, one can use the rate equation to determine the rate of the state variables. If, for every P, there is a unique solution of the state variable rate by (10) and (11), then we call the system normal at the neighborhood of the given state. Equivalently, one may state that when the system is normal, there is only zero solution to the rate equation if the loading rate is zero. On the other hand, if there is more than one solution for a given loading rate, then we say system is in a critical condition at the given state. In other words, when a system is critical, there can be a non-zero solution to the rate equation even when the loading rate is zero. Therefore, one can study the behavior of CCM by study the behavior of its rate equation.

4. UNLOADING OF A COHESIVE CRACK

A cohesive crack is called in loading condition, if its crack opening displacement increases, otherwise it is called in unloading condition. In the preceding discussion, it was assumed that all cracks are in the loading condition. For the purpose of studying the structural behavior such as its load-deflection curve, it is usually sufficient for CCM to deal only with loading crack. This is perhaps the main reason why little is studied about the unloading behavior of CCM. When there are more than one cracks, it is possible that some cracks are in loading condition while the other is in unloading condition, even when the applied load P increases.

When a cohesive crack is unloading, the cohesive stress does not follow function g to increase when the crack opening displacement w decreases, because the material in the process zone is partially damaged and the damage cannot be reversed by crack closure. While the actual behavior of the unloading stress-displacement relation is, obviously, a subject of experimental study, a simple relation is proposed as shown in Fig. 2. Upon closing, the stress reduces linearly back to the origin. If the crack is reopened, the stress increases along the unloading line until the softening curve g is reached, then it decreases again following the softening curve g. This relation implies that the material can close perfectly to its original position. In reality the fractured surfaces are rough, the crack opening displacement may be unable to return to zero. Once may need to use more sophisticated unloading relation to describe the crack closure. However, the precise nature of the unloading is beyond the scope of this paper.

A loading crack implies a propagating crack. When a crack propagates, one requires that the total stress intensity factor be zero, as is done in (4) or (5). Since the position of the process zone tip is unknown for a propagating crack, the condition of zero stress intensity factor can be viewed as a condition to determine the crack tip position. On the other hand, if a crack is unloading, it no longer propagates. The process zone tip position of an unloading crack is thus not a variable, and its correspondent crack tip equation becomes redundant. Consequently, whenever a crack is in unloading condition, the corresponding component in the crack tip eqns (5) must be discarded so that the number of equations and the number of unknowns match. The rate form of the crack tip equation must also be modified accordingly

$$\frac{\mathrm{d}}{\mathrm{d}t}K_{i'} = k_{i'}^P \dot{P} - \int_{a_j}^{b_j} k_{i'j}^\sigma(x)\dot{\sigma}_j(x)\,\mathrm{d}x + \dot{b}_{m'} \frac{\partial k_{i'}^P}{\partial b_{m'}} P - \dot{b}_{m'} \frac{\partial}{\partial b_{m'}}\int_{a_j}^{b_j} k_{i'j}^\sigma(x)\sigma_j(x)\,\mathrm{d}x = 0 \tag{12}$$

where a subscript with a prime denotes that the components corresponding to unloading cracks are excluded. For instance, suppose there are two cracks under consideration. Crack one is loading while crack two is unloading. Then the rate form of the crack tip equation for crack two must be discarded and the rate form of crack tip equation for crack one becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}K_1 = k_1^P\dot{P} - \int_{a_1}^{b_1} k_{11}^\sigma(x)\dot{\sigma}_1(x)\,\mathrm{d}x + \int_{a_2}^{b_2} k_{12}^\sigma(x)\dot{\sigma}_2(x)\,\mathrm{d}x$$
$$+\dot{b}_1\left[\frac{\partial k_1^P}{\partial b_1}P - \frac{\partial}{\partial b_1}\int_{a_1}^{b_1} k_{11}^\sigma(x)\sigma_1(x)\,\mathrm{d}x - \frac{\partial}{\partial b_1}\int_{a_2}^{b_2} k_{12}^\sigma(x)\sigma_2(x)\,\mathrm{d}x\right] = 0. \quad (13)$$

Furthermore, the last term in (10) is always zero regardless whether a crack is loading or unloading, because for a loading crack we have $\dot{b}_m > 0$ but $K_m = 0$, whereas for an unloading crack we have $\dot{b}_m = 0$ although K_m may not be zero. In either case, the product $\dot{b}_m K_m$ is always zero. As a result, the rate eqn (10) can always be simplified as

$$\frac{\mathrm{d}}{\mathrm{d}t}G_i = g'\dot{\sigma}_i(x_i) + \int_{a_j}^{b_j} C_{ij}^{\sigma\sigma}(x_i, x)\dot{\sigma}_j(x)\,\mathrm{d}x - C_i^{\sigma P}(x_i)\dot{P} = 0. \quad (14)$$

In other words, the rate form of the process zone equation is independent of $\dot{b}_m$. One may take advantage of this special structure of the rate equation in studying the general behavior of CCM. We will come back to this point later.

5. STIFFNESS FORMULATION

Sometimes, it is more convenient to work with the stiffness formulation, in which the influence functions are stiffness functions and the crack opening displacement is the basic unknown. The stiffness function can be defined as the inverse of the compliance function in the following manner.

Denote by u the load-line displacement, and one can write the elastic relations in terms of stiffness function as

$$\sigma_i(x_i) = -\int_{a_j}^{b_j} S_{ij}^{ww}(x_i, x)w_j(x)\,\mathrm{d}x + S_i^{wu}(x_i)u \quad (15)$$

$$P = -\int_{a_j}^{b_j} S_j^{uw}(x)\,\mathrm{d}x + S^{uu}u. \quad (16)$$

The stiffness functions and the compliance functions are connected to each other by the following four reciprocal relations:

$$\int_{a_m}^{b_m} S_{im}^{ww}(x_i, t)C_{mj}^{\sigma\sigma}(t, x_j)\,\mathrm{d}t + C_i^{\sigma P}(x_i)S_j^{uw}(x_j) = \delta_{ij}\delta(x_i - x_j) \quad (17)$$

where δ_{ij} is the Kronecker-delta function and $\delta(x-y)$ is the Dirac-delta function, and

$$\int_{a_m}^{b_m} S_m^{wu}(t)C_m^{P\sigma}(t)\,\mathrm{d}t + C^{PP}S^{uu} = 1 \quad (18)$$

where C^{PP} = load-line displacement due to a unit load P, and

$$\int_{a_m}^{b_m} C_{im}^{\sigma\sigma}(x_i, t) S_m^{wu}(t)\,\mathrm{d}t + C_i^{\sigma P} S^{uu} = 0 \tag{19}$$

$$\int_{a_m}^{b_m} C_m^{P\sigma}(t) S_{mj}^{ww}(t, x_j)\,\mathrm{d}t + C^{PP} S_j^{uw}(x_j) = 0. \tag{20}$$

These four relations provide enough equations to determine the stiffness functions for given compliance functions, or vice versa.

The softening law (2) can be inverted and written as

$$\sigma = f(w). \tag{21}$$

Thus the process zone equation can be expressed as

$$G_i = f[w_i(x_i)] + \int_{a_j}^{b_j} S_{ij}^{ww}(x_i, x') w_j(x')\,\mathrm{d}'x - S_i^{wu}(x_i) u = 0 \tag{22}$$

where the loading parameter is load-line displacement, which corresponds to the case that the structure is under displacement control. If the structure is under load control, then the second equation of (15) must be utilized together with (21) to eliminate u. Such a substitution of u with P also modifies the stiffness function. However, we will not pursue the detail in this paper.

The crack tip equation can be written as

$$K_i = k_i^u u - \int_{a_j}^{b_j} k_{ij}^w(x) w_j(x)\,\mathrm{d}x = 0 \tag{23}$$

where k_i^u = stress intensity factor in the tip of crack i due to a unit load-line displacement u, and k_{ij}^w = stress intensity factor in the tip of crack i due to a unit crack opening displacement at x_j. It is noted that most stress intensity factor formulae are given in the form of given unit force, rather than give unit displacement. Fortunately, these two types of stress intensity factor formulae can be related by the following equations:

$$k_{ij}^w(x_j) = -\int_{a_m}^{b_m} S_{mj}^{ww}(x, x_j) k_{im}^{\sigma}(x)\,\mathrm{d}x + S_j^{uw}(x_j) k_i^P \tag{24}$$

and

$$k_i^u = -\int_{a_j}^{b_j} S_j^{wu}(x) k_{ij}^{\sigma}(x)\,\mathrm{d}x + S^{uu} k_i^P. \tag{25}$$

It is easy to verify that the derivatives of the stiffness function can be expressed in terms of stress intensity $k_{ij}^w(x)$ and k_i^u as follows

$$\frac{\partial S_{ij}^{ww}(x_i, x_j)}{\partial b_m} = -\frac{2}{E'} k_{mi}^w(x_i) k_{mj}^w(x_j)\,; \quad \frac{\partial S_i^{wu}(x_i)}{\partial b_m} = -\frac{2}{E'} k_{mi}^w(x_i) k_m^u \tag{26}$$

which are entirely parallel to (7) apart from a negative sign. Again, there is no summation implied over the subscript m.

With (26), the rate form of the process zone equation (22) can be written as

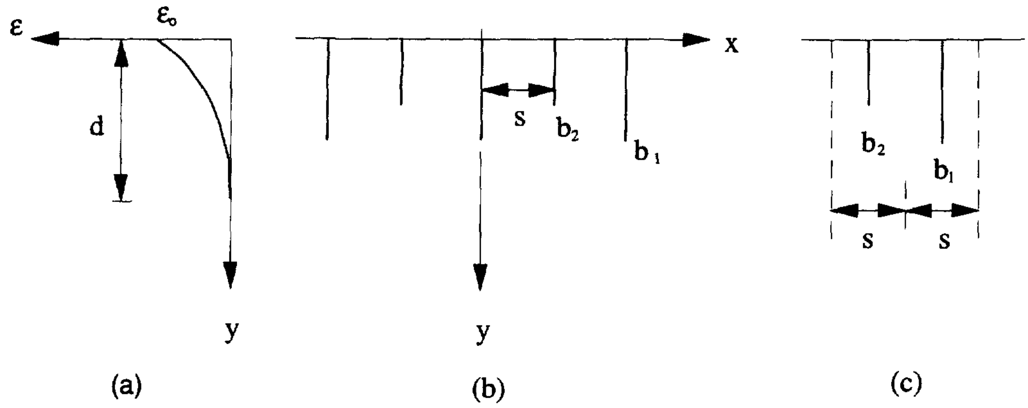

Fig. 3. (a) Shrinkage stress distribution; (b) geometry definition; (c) a unit cell with two cracks.

$$\frac{\mathrm{d}}{\mathrm{d}t}G_i = f'\dot{w}_i(x_i) + \int_{a_j}^{b_j} S_{ij}^{ww}(x_i, x')\dot{w}_j(x')\,\mathrm{d}'x - S_i^{wu}(x_i)\dot{u} = 0 \tag{27}$$

which is still independent of b_m. The rate form of the crack tip equation is the same as (11) except the σ should be replaced with w, and P should be replaced with u. If there are unloading cracks, the rate equations should be modified accordingly. In other words, f' should be calculated according to unloading curve, and the corresponding crack tip equations should be discarded.

6. PARALLEL CRACKS ON THE SURFACE OF A HALF PLANE

The theory described previously is applicable to a structure with any number of cracks. In what follows we will discuss a problem with two interactive cracks. Furthermore, we assume the softening law is a linear function in the sense that the derivative f' (or g') is a constant for loading cracks. Under this assumption, (27) is independent of the crack opening displacement.

Imaging that the half plane is loaded by shrinkage strain near the surface with a distribution shown in Fig. 3(a). Physically, such an initial strain may be caused by surface cooling or surface drying. The crack is uniformly spaced along the surface with a distance s which is assumed to be a parameter, see Fig. 3(b). We further assume that initial crack length $a_i = 0$ and every other crack has the same length. Thus, the problem can break down into a unit cell of width $2s$ (see Fig. 3(c)) with two cracks.

The initial strain causes initial stress which is tensile. The stress distribution is represented by the function $\lambda F(x; d)$, where $F(0, d) = 1$ and $F(d, d) = 0$, see Fig. 3(a), where λ = maximum stress and d = loading depth. Both d and λ can be treated as loading parameters, but only one should be left as a free variable. For a specific physical process, both d and λ are functions of time, and we can use time as the loading parameter. In this paper, λ is used as loading parameter for simplicity, although the analysis is equally applicable to other possible selection of loading parameters.

When cracks develop, this tensile stress becomes zero on the crack surfaces. To satisfy this boundary condition, a compressive stress of the same distribution is applied to the crack surfaces. Such a compressive stress is the driving force to keep a crack open. The process zone equation can be written as

$$G_i = f[w_i(x_i)] + \int_0^{b_j} S_{ij}^{ww}(x_i, x')w_j(x')\,\mathrm{d}'x - F(x_i; d)\lambda = 0 \quad i = 1, 2 \tag{28}$$

which is slightly different from (22) because the problem discussed here is load controlled, and the loading stress distribution can be solved in advance.

The crack tip equation is also slightly different, which can be expressed as

$$K_i = -\int_0^{b_j} k_{ij}^w(x) w_j(x)\,\mathrm{d}x = 0 \quad i = 1,2. \tag{29}$$

The crack tip equation can also be expressed as

$$K_i = \int_0^{b_j} k_{ij}^\sigma(x)(\lambda F(x_j; d) - \sigma_j(x))\,\mathrm{d}x = 0 \quad i = 1,2 \tag{30}$$

where σ_j is the cohesive stresses. This equation has a clearer physical meaning. It requires that the stress intensity factors caused by the load must cancel the stress intensity factor caused by the cohesive stress. Mathematically, these two expressions are equivalent because, in this case, the two different types of stress intensity factors are related by the equation

$$k_{ij}^w(x_j) = -\int_0^{b_m} S_{mj}^{ww}(x, x_j) k_{im}^\sigma(x)\,\mathrm{d}x. \tag{31}$$

To solve the problem numerically, we use the following procedure:

(1) Select a value of b_1.

(2) For a guessed value of b_2, ($<b_1$) solve the equations $G_i = 0$ ($i = 1,2$) and $K_1 = 0$ simultaneously for w_i ($i = 1,2$) and the loading parameter d, using Newton's method.

(3) Using Newton's method to get an updated value of b_2 through the equation $K_2 = 0$ based on the solution obtained in step (2), go back to (2). Repeat this loop until all the equations are satisfied.

(4) Choose another value of b_1 and repeat the whole procedure.

The actual computation is based on a set of Cauchy's singular integral equations in which the dislocation density functions are the basic unknown together with the crack lengths. In this formulation, the crack length can vary continuously, as contrast to the case of finite element method or boundary element method where the crack length can only assume values compatible with the given element mesh. A similar formulation was utilized by Nemat-Nasser *et al.* (1978) to solve the problem of interactive cracks by linear elastic fracture mechanics.

7. BEHAVIOR OF THE SOLUTION: BIFURCATION AND MAXIMUM LOAD

Although b_1 is treated as a parameter while b_2 is treated as an unknown variable in our solution procedure, a symmetric solution, in which the two cracks are of equal length and the two crack displacements are the same, can always be found. In other words, even though b_2 is deliberately set to be less than b_1, the final value of b_2 always converges to b_1. For a given load depth d, the loading parameter λ vs crack length is plotted in Fig. 4 as the top curve. The crack system is at critical condition when the applied load is at its maximum, and the corresponding crack length is denoted as c_{max}. It is noted that the crack mouth opening is always less than the threshold value w_c at the maximum load. As stated previously, the homogeneous rate equations have a non-zero solution for the displacement rate and crack length rate when $b_i = c_{max}$. The homogeneous rate equations become

$$\frac{\mathrm{d}}{\mathrm{d}t} G_i = f'\dot{w}_i(x_i) + \int_0^{b_j} S_{ij}^w(x_i, x')\dot{w}_j(x')\,\mathrm{d}'x = 0 \tag{32}$$

and

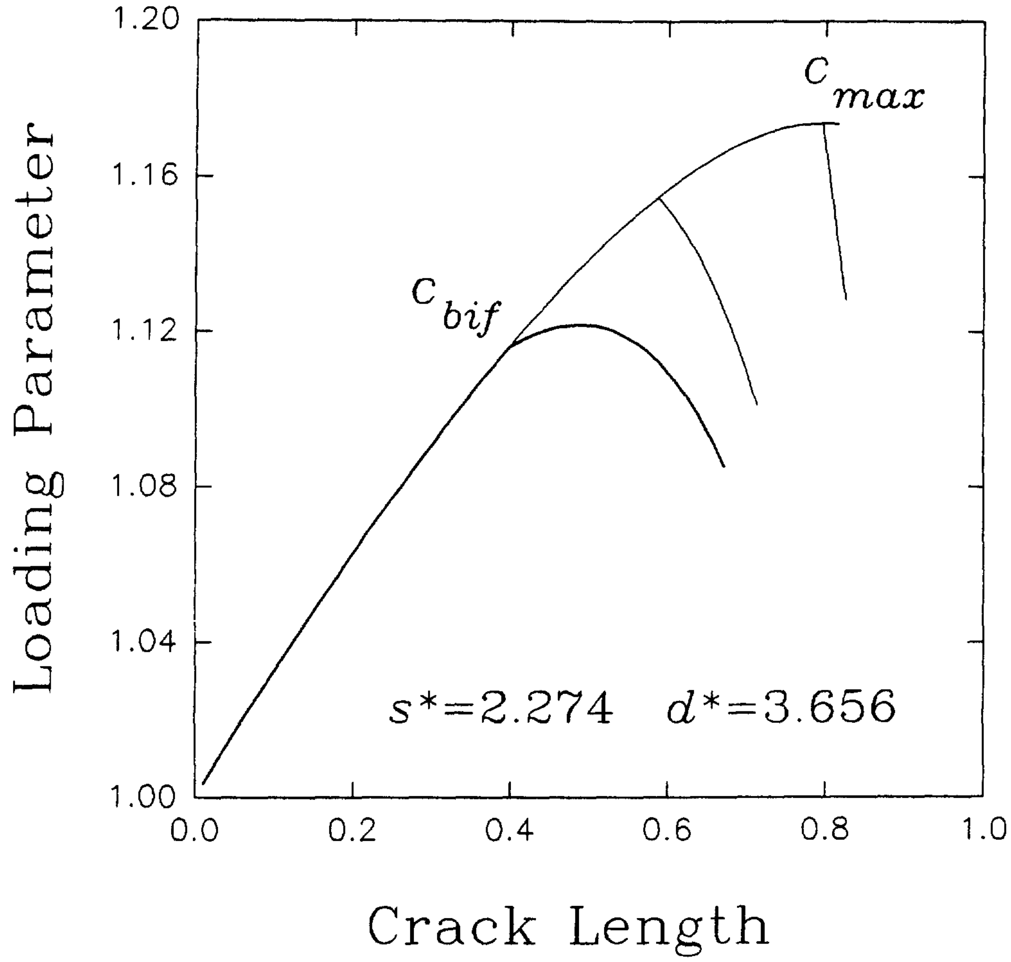

Fig. 4. The applied load as a function of crack length.

$$\frac{\mathrm{d}}{\mathrm{d}t}K_i = -\int_0^{b_j} k_{ij}^w(x)\dot{w}_j(x)\,\mathrm{d}x + \dot{b}_m \frac{\partial}{\partial b_m} - \int_0^{b_j} k_{ij}^w(x) w_j(x)\,\mathrm{d}x = 0. \tag{33}$$

Because of the special structure of the rate equations, the homogeneous rate equations admit non-zero solution if (32) admit non-zero solution. This is because once non-zero displacement rate can be found from (32), non-zero crack length rate can always be found from (33). Thus, the task of finding a non-zero solution from the homogeneous rate equation is simplified. Furthermore, since the softening law is linear, $f' = -f_t/w_c$ is a constant. For convenience, let us introduce non-dimensional variables as follows

$$\bar{S}_{ij}^{ww} = \frac{S_{ij}^{ww}}{E'}; \quad \beta_j = \frac{b_j}{b}; \quad b^* = \frac{b}{l_{ch}} \tag{34}$$

where $l_{ch} = EG_f/f_t^2$ is called the characteristic length of the material. For linear softening law, the fracture energy $G_f = f_t w_c/2$. The non-dimensional stiffness functions depend only on the geometry of the problem. In the present problem, they are functions of b_j/s. Since we are discussing the critical condition of the symmetric solution, we choose $b = b_1 = b_2$. As a result, (32) can be written as

$$2b^* \int_0^1 \bar{S}_{ij}^{ww}(x_i, x')\dot{w}_j(x')\,\mathrm{d}'x = \dot{w}_i(x_i). \tag{35}$$

For given crack spacing s, finding b such that (35) has a non-zero solution is a difficult

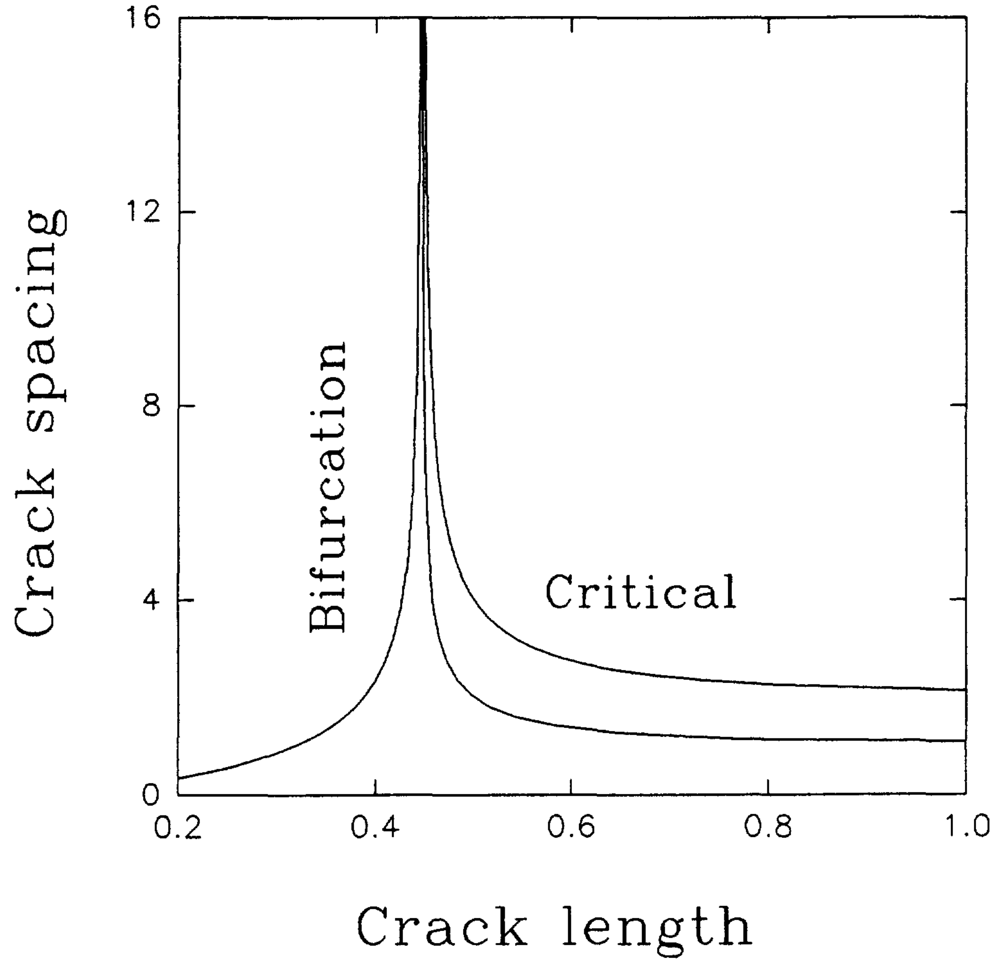

Fig. 5. Critical crack lengths as a function of crack spacing.

task. However, the problem is substantially simplified if (35) is viewed as an eigenvalue problem for given b/s. In particular, we are interested only in the smallest eigenvalue of $2b^*$. Once b^* is known as a function of b/s, then b^* can also be expressed as a function of $s^* = s/l_{ch}$. Such curves are plotted in Fig. 5. The right curve corresponds to the eigenfunction $w_1 = w_2$ and the left curve corresponds to $w_1 = -w_2$. The curve in the middle will be discussed later.

The right curve corresponds to the condition that the applied load is maximized when the crack length is equal to the given value for a given crack spacing. In fact, the maximum load can be calculated directly using the eigenvalue solution. The process zone equation can be recast into non-dimensional form as

$$1-\bar{w}_i(x_i)+\int_0^{b_j}\bar{S}_{ij}^{ww}(x_i,x')\bar{w}_j(x')\,\mathrm{d}'x-F(x_i;d)\bar{\lambda}=0 \tag{36}$$

where

$$\bar{w}_j=\frac{w_j}{w_c}\quad \bar{\lambda}=\frac{\lambda}{f_t}. \tag{37}$$

Multiplying the eigenfunction by (36) and integrating with respect to x_i, we find that the loading parameter can be computed from the expression

$$\frac{\lambda}{f_t} = \frac{\int_0^1 \dot{w}_1(x)\,\mathrm{d}x}{\int_0^1 \dot{w}_1(x)F(x_i\,;d)\,\mathrm{d}x_i}. \tag{38}$$

In deriving this equation, we have utilized the critical condition (35) and the symmetric property of the stiffness functions, as well as the symmetric condition of the eigenfunction. This solution is useful if one need to know the maximum load as a function of crack spacing.

The left curve in Fig. 5 obviously is not related to the maximum load of the symmetric solution. Since $w_1 = -w_2$ for the lower curve, the corresponding load level cannot be calculated from (38). Actually, the lower curve corresponds to the bifurcation point for a given crack spacing. The symmetric solution is the unique solution if b_1 is less than the bifurcation length and both cracks are under loading condition. However, the symmetric solution is no longer the only solution if b_1 is greater than the bifurcation length. When the symmetric solution passes the bifurcation length, crack two may unload, leave only crack one to grow in response to the loading. Consequently, the solution loses its symmetry. The load required to propagate only one crack is usually less than that of a symmetric solution. Although both solutions are stable, the asymmetric solution is the most likely solution that is actually followed.

8. BEHAVIOR OF THE SOLUTION: MAXIMUM LOAD WITH ONE CRACK UNLOADING

Once crack two starts unloading, the corresponding crack tip equation is disregarded. If the crack opening displacement at the bifurcation point is denoted as w_2^*, then the unloading stress-displacement equation is defined as $\sigma = f(w^*)w/w^*$. The solution procedure is modified because one needs not to worry about K_2 any more. It is also checked to see if there is crack closure in crack two. For the range calculated, there is no crack closure found (this of course does not exclude the possibility of crack closure when crack one is elongated further). The load parameter as a function of the active crack length is also shown in Fig. 4 as the curve in the bottom. Compared with the curve on the top, which is the load required if two cracks grow simultaneously, one can easily understand why one crack unloading is more likely to happen in the real situation.

With one of the cracks unloading and the other growing, the load reaches a maximum value, which corresponds to yet another critical condition. The critical condition can again be characterized by the homogeneous rate eqn (32), in an expanded form, as

$$b_1^* \int_0^{b_1} S_{11}^{ww}(x_1, x')\dot{w}_1(x')\,\mathrm{d}'x + b_2^* \int_0^{b_2} S_{12}^{ww}(x_1, x')\dot{w}_2(x')\dot{w}_2(x')\,\mathrm{d}'x = \dot{w}_1(x_1)/2$$

$$b_1^* \int_0^{b_1} S_{21}^{ww}(x_2, x')\dot{w}_1(x')\,\mathrm{d}'x + b_2^* \int_0^{b_2} S_{22}^{ww}(x_2, x')\dot{w}_2(x')\,\mathrm{d}'x = -\frac{f(w^*)w_c}{2f_t w^*}\dot{w}_2(x_2). \tag{39}$$

When the stability limit condition is satisfied, there exists a non-zero solution of crack opening rate to this homogeneous equations. This equation cannot be simplified into a linear eigenvalue problem, because there is negative sign in the right hand side of the second equation of (39). To ensure a correct numerical solution, one need to solve the crack opening rate of crack two in terms of the crack opening rate of crack one from the second equation for an assumed value of b_1^*, then eliminate the crack opening rate of crack two in the first equation to obtain a well-posed eigenvalue problem. Once this eigenvalue problem is solved, the corrected value of b_1^* should be substituted back to the second equation, and the whole procedure iterates until the value b_1^* is converged. This procedure usually takes only very few steps of iteration, typically 2 or 3 for 4-digit accuracy in b_1^*, since the singularity condition is mainly determined by crack one and the influence of crack two on crack one is not significant.

The smallest eigenvalue of the singularity condition yields yet another critical crack length for crack one, which is the curve plotted in the middle of Fig. 5. It is interesting to note that this curve is virtually the same as the right curve if its crack spacing b^* is multiplied by a factor of 2. The corresponding maximum load can also be obtained by multiplying the eigenfunction to the equilibrium equations and integrating along the crack process zone

$$\frac{\lambda}{f_t} = \frac{b_1^* \int_0^1 \dot{w}_1(x)\,\mathrm{d}x}{b_1^* \int_0^1 \dot{w}_1(x)F(x;d)\,\mathrm{d}x + b_2^* \int_0^1 \dot{w}_2(x)F(x;d)\,\mathrm{d}x}. \tag{40}$$

So far in this section we have assumed that crack two starts to unloading right at the point of bifurcation. Although this is most likely the case, it does not have to be this way. Actually, b_2 can be any value between c_{bif} and c_{max}, the corresponding curves are also shown in Fig. 4. In this sense, the solution is not unique once the crack lengths exceed the bifurcation point. This is very similar to the case of the buckling of an elastoplastic column (Bazant and Cedolin, 1991): once the applied load exceeds the tangent modulus load, there can be more than one solution bifurcating from the original symmetric solution. More detailed discussion on the solution behaviors as well as the consequences of the interaction of multiple interactive cohesive cracks in connection with static crack initiation problem will be published in another paper.

9. DISCUSSIONS AND CONCLUSIONS

The mechanical interaction of cracks is a highly nonlinear problem. In addition to stability limit, there is also bifurcation of crack growth patterns. Although the interaction problem has been discussed in the framework of linear elastic fracture mechanics, it does not seem to have been studied in the context of the nonlinear cohesive crack model.

With linear fracture mechanics, the emphasis is on the crack interaction during the stage of crack growth. With the cohesive crack model, one can zoom in on the crack interaction during the stage of crack initiation. In the example problem studied previously, both bifurcation and stability limit occur before crack mouth opening exceeds its threshold value. In other words, cracks have not started to propagate yet. These phenomena, therefore, cannot be properly described by using linear elastic fracture mechanics, because in linear fracture mechanics, and in this matter, elastoplastic fracture mechanics as well, a material can either be cracked or not cracked. To capture the gradual process of material deterioration, the theory of cohesive crack model should be used.

In this paper, the theory cohesive crack model with multiple cracks are formulated with compliance functions and stiffness functions, with stipulations on unloading cracks. The rate equations of the cohesive crack model are derived. It is demonstrated that the critical condition can be determined from the considering only the homogeneous rate equation of the process zone equation. In this way, the task to determine the critical crack length becomes somewhat easier. If the softening law can be assumed linear, the eigenvalue problem is linear, and the maximum loads can be determined directly from the eigenfunction. Without the knowledge of the overall structure of the solution behavior, which can only be obtained through a stability analysis, a blind solution of cohesive crack model with more than one crack can be very misleading.

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